Use of the Curvature Method To Determine True Vertical Reservoir Thickness

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Introduction
Net pay is one of the most important factors in determining equity participation formulas. In the past, it was picked directly from the logs of reasonably vertical holes with relatively negligible error. In directionally drilled wells, net pay must be selected by correcting the observed log thickness of a zone to the value that would have been logged if the well had penetrated the zone vertically through the point it pierced the top of the bed. These corrections must take into account the borehole inclination as well as the dip and strike of the formation. They may be made using the true coordinates of the points at which the top and bottom of the pay zone are penetrated by the well. These coordinates can be obtained by the Tangential (straight line) method of directional survey interpretation, which in certain cases has doubtful accuracy. In those instances when the segments between survey stations are curved, the Tangential method will cause errors that, because of their cumulative effects, may become substantial.

To correct for the effects of these errors, Wilson presented an improved method of directional survey computation, which he called the “Radius of Curvature”. He derived the necessary equations for those cases in which the curvature of the wellbore was either on a vertical or on a horizontal plane, pointing out that the Tangential method was materially a different interpretive technique.

Wilson’s solutions may be generalized. A general set of equations of the Curvature method of directional survey interpretation can be shown to converge into the equations of the Tangential method when the borehole curvature either is zero or approaches zero. The purpose of this paper is to derive an analytical expression for true vertical formation thickness when a bed of known dip and strike is pierced by a slanted well. Since the accuracy of these calculations is dependent upon the true location of points on the wellbore, the Curvature method affords at this time a computational technique that is superior to those existing heretofore. The wide availability of computers makes the application of the equations an easy task.

Directional Survey Interpretation
For every survey station, three items of information are recorded by companies engaged in running directional surveys: (1) the depth at which the instrument was stopped; (2) the drift (inclination) of its axis off the vertical; and (3) the drift direction on a horizontal plane.

Two sources of possible error are immediately apparent. First, the tool may not have been centralized; that is, its axis may not have coincided with that of the wellbore. Second, the measured depth may have been in error because of cable stretch or because the cable was not centered in the wellbore. Other sources of error inherent in particular recording instruments may also become apparent. However, even if these sources of error are ignored, it is possible to introduce a computational error during the interpretation of the survey, if it is assumed that survey points are connected by straight lines. The accuracy of this method

A generalized curvature method of directional survey interpretation can be shown to converge into the Tangential method for curvatures close to zero. True vertical thickness corrections may be made based on directional surveys, taking into account the slant of the hole and the dip of the formation.
increases as the spacing between survey points decreases, but very close spacing may not be economically feasible.

To improve the accuracy of computation, Wilson introduced the Radius of Curvature method of interpretation, basing it on the assumption that the wellbore has a constant curvature either on a horizontal or on a vertical plane. A measure of the curvature is obtained from the change in drift angle and direction from one survey station to the next. Wilson's analysis may be expanded into sets of solutions that cover those instances when the curvature of the wellbore is not on a vertical plane. From the observed drift angles (φ) and their directions (θ) at two consecutive survey points, a and b, four possible cases may be envisioned:

**Case 1:** (φb − φa) ≠ 0; (θb − θa) ≠ 0
**Case 2:** (φb − φa) ≠ 0; (θb − θa) = 0
**Case 3:** (φb − φa) = 0; (θb − θa) ≠ 0
**Case 4:** (φb − φa) = 0; (θb − θa) = 0

The derivation of the solutions for each of these cases is presented in Appendix A. The solutions for each of the four cases in terms of incremental distances in the x, y and z directions are as follows:

**Cases 1 and 2. φ ≠ φ**

\[ z_b - z_a = \frac{L_b - L_a}{\phi_b - \phi_a} (\sin \phi_b - \sin \phi_a) \]  

**Case 1**

\[ \theta_b \neq \theta_a [ -\pi \leq (\theta_b - \theta_a) \leq \pi] \]

\[ x_b - x_a = \frac{(L_b - L_a)(\cos \phi_b - \cos \phi_a)(\cos \theta_b - \cos \theta_a)}{(\phi_b - \phi_a)(\theta_b - \theta_a)} \]  

**Cases 3 and 4. φ = φ**

\[ z_b - z_a = (L_b - L_a)(\cos \phi_a) \]  

**Case 3**

\[ \theta_b \neq \theta_a [ -\pi \leq (\theta_b - \theta_a) \leq \pi] \]

\[ x_b - x_a = \frac{(L_b - L_a)(\sin \phi_a)(\cos \theta_b - \cos \theta_a)}{(\theta_b - \theta_a)} \]

\[ y_b - y_a = \frac{(L_b - L_a)(\sin \phi_a)(\sin \theta_b - \sin \theta_a)}{(\theta_b - \theta_a)} \]  

**Case 4**

\[ \theta_b = \theta_a \]

\[ x_b - x_a = (L_b - L_a)(\sin \phi_a)(\sin \theta_a) \]

\[ y_b - y_a = (L_b - L_a)(\sin \phi_a)(\cos \theta_a) \] . . . . (10)

It should be emphasized that the expected change in drift direction azimuth may introduce a sizable error if it is not treated adequately. This change is limited to ±180° (or ±π radians), since any change greater than this would imply a change in the opposite direction. For example, if θa = 350° and θb = 2°, the resulting (θb − θa) is 12° and not −348°. This is a reasonable assumption as long as the drilled section is short and as long as it is physically impossible for the drill pipe to bend −348°.

The expressions in Eqs. 6, 9, and 10 are those used by the Tangential method of directional survey interpretation. These are applicable only when the wellbore curvature is zero. In practice, when the curvature approaches zero, the error introduced in each segment by straight-line representation is rather small. Fig. 1 shows that an error of only 1 ft per 100 ft of depth is introduced by using the Tangential method in a borehole segment with a curvature of 10°/100 ft. Insignificant as this error may seem, its cumulative effect over the entire length of the borehole may result in drastic discrepancies as to the location of the bottom of the hole. It may be contended that a 1-percent error correction is well beyond the accuracy of the recording instrument. Fig. 1 shows the variance that can be expected for instrument inclination errors of ±0.5°. This range of error is within the maximum range in most instruments, yet it does not change the significance of the error appreciably. As a further analysis, it is possible to extend the work done by Walstrom et al. to determine the degree of certainty that can be associated with the computed bottom-hole locations for a given tool. This task, however, is beyond the scope of this paper.

It should be recognized that other approaches to directional survey interpretation are still open. For instance, instead of a constant wellbore curvature between two survey points, it may be useful to consider a changing curvature and analyze three or four points at a time. The possible improvements in accuracy to be gained with these or other, more sophisticated, methods must be considered in the light of possible sources of error. Any interpretive technique must take into account its inherent model errors as well as the limitations of the recording instruments.

**Correction of Vertical Thickness**

The problem of correcting for true vertical thickness arises when it is desirable to compute net pay from apparent log intervals recorded on directionally drilled holes. The correct net pay is used in many reservoir engineering calculations and in determining equity participation formulas. It must be computed taking into account the effect of the inclination of the wellbore as well as the strike and dip of the reservoir beds.

Eq. 11 gives an expression for true vertical thickness derived in detail in Appendix B.

\[ h_t = (z_b - z_a) - \tan \alpha_d (y_b - y_a) \cos \gamma \]

\[ + (x_d - x_a) \sin \gamma \] . . . . (11)
The subscripts \( o \) and \( d \) indicate the top and bottom of the bed, respectively. Fig. 2 shows the computed true vertical thickness of a pay zone where the well has a constant 45° inclination and a N 45° E bearing, as a function of the formation dip and its direction. The observed log thickness of this hypothetical well is a constant 100 ft. The dip angle was changed from 0° to 10°, and the azimuth of its direction from 0° to 360°. It should be noted that only when the direction of the well parallels the strike of the bed, true vertical thickness is independent of bed dip. In Fig. 2, this occurred at 135° and again at 315°. Obviously, when the dip is zero the observed log thickness must be corrected only by the inclination of the wellbore.

Another observation can be made on Fig. 2. If the abscissa is expressed as degrees clockwise from the direction of the well, the same curves apply to any direction the well may take. For example, consider a well that has an observed 100 ft of log interval, a 45° inclination and a S 30° E direction. Assume that the formation being penetrated is dipping 10° with an 80° azimuth. The dip azimuth is therefore 290° ahead of the direction of the well. Starting at 45° on the abscissa of Fig. 2, the 290° would correspond to 335°. At this point the corrected vertical thickness would be approximately 66.3 ft.

To apply Eq. 11 one must know the coordinates \( x_o, y_o, z_o, x_d, y_d, z_d \). These points can be obtained from an analysis of directional survey data by expanding the curvature method to points located within two consecutive survey stations. This procedure, shown in Appendix B, involves obtaining the inclination and the directional azimuth angles \( \phi \) and \( \theta \) for points \( o \) and \( d \). Once these angles have been determined, it is possible to compute the true coordinates at those points and apply Eq. 11.

When the directional survey points are not too far apart, there is an alternate method that may be followed to determine the location of points \( o \) and \( d \). This method may be followed in most cases without introducing appreciable error, if the curvature is not excessive and if the true coordinates of the survey points \( a \) and \( b \) have been fixed by the Curvature method of analysis. In this case it may be sufficient to obtain the incremental coordinates by proportions. Thus:

\[
\begin{align*}
  x_d - x_o & = \frac{L_d - L_o}{L_b - L_a} (x_b - x_o) \quad (12) \\
  y_d - y_o & = \frac{L_d - L_o}{L_b - L_a} (y_b - y_o) \quad (13) \\
  z_d - z_o & = \frac{L_d - L_o}{L_b - L_a} (z_b - z_o) \quad (14)
\end{align*}
\]

Eqs. 12 through 14 may be useful for hand calculations.

It may also be desirable to compute \( h_t \) using the average inclination and direction of the borehole through the pay interval. In this case Eq. 11 may be rewritten:

\[
h_t = (L_a - L_o) \sqrt{\cos \overline{\phi} \tan \overline{\phi} \sin \overline{\phi} \cos (\overline{\theta} - \gamma)},
\]

where

\[
\overline{\phi} = \frac{\phi_d + \phi_o}{2} = \frac{\phi_b + \phi_a}{2} \quad (16)
\]

\[
\overline{\theta} = \frac{\theta_d + \theta_o}{2} = \frac{\theta_b + \theta_a}{2} \quad (17)
\]

Eq. 17 should be used with care, realizing the inherent errors that may be committed by averaging angles. Obviously the average of 359° and 1° is not 180° (due south) but 0° (due north).
Eq. A-4 gives the incremental true vertical depth, applicable when $\theta_b$ is different from $\theta_a$.

To derive expressions for departure, assume that the projection of $ab$ on a horizontal plane has a constant radius of curvature; then, by definition,

$$\frac{d\theta}{ds} = \frac{\theta_b - \theta_a}{s_b - s_a} = \text{constant} \quad \ldots \quad (A-5)$$

This assumption contains the condition that

$$-\pi \leq (\theta_b - \theta_a) \leq \pi, \quad \ldots \quad (A-6)$$

since the change in $\theta$ is not expected to exceed 180°. From inspection of Fig. 3 we can write

$$\sin \phi = \frac{ds}{dL} \quad \ldots \quad (A-7)$$

$$\sin \theta = \frac{dx}{ds} \quad \ldots \quad (A-8)$$

$$\cos \theta = \frac{dy}{ds}. \quad \ldots \quad (A-9)$$

Also, from calculus we have

$$\frac{ds}{d\phi} = \frac{dL}{d\phi} \quad \frac{ds}{d\phi} \quad \ldots \quad (A-10)$$

Substitution of Eqs. A-2 and A-7 into Eq. A-10 yields

$$\frac{ds}{d\phi} = \left(\frac{L_b - L_a}{\phi_b - \phi_a}\right) \sin \phi.$$ \quad \ldots \quad (A-11)

Separating variables and integrating,

$$\int_a^b ds = \left(\frac{L_b - L_a}{\phi_b - \phi_a}\right) \int_{\phi_a}^{\phi_b} \sin \phi d\phi$$

$$s_b - s_a = \frac{L_b - L_a}{\phi_b - \phi_a} \left(\cos \phi_b - \cos \phi_a\right). \quad \ldots \quad (A-11)$$

Also, by calculus,

$$\frac{d\theta}{ds} = \frac{dx}{ds} \quad \frac{d\theta}{ds} \quad \ldots \quad (A-12)$$

Substitution of Eqs. A-5 and A-8 into Eq. A-12 yields

$$\frac{\theta_b - \theta_a}{s_b - s_a} = \sin \theta \quad \frac{d\theta}{dx} \quad \ldots \quad (A-12)$$

Separating variables and integrating,

$$\int_a^b dz = \left(\frac{L_b - L_a}{\phi_b - \phi_a}\right) \int_{\phi_a}^{\phi_b} \cos \phi d\phi$$

$$z_b - z_a = \frac{L_b - L_a}{\phi_b - \phi_a} \left(\sin \phi_b - \sin \phi_a\right). \quad \ldots \quad (A-4)$$
\[ b \]
\[ \int_a^b dx = \left( \frac{s_b - s_a}{\theta_b - \theta_a} \right) \sin \theta d\theta \]
\[ x_b - x_a = \left( \frac{s_b - s_a}{\theta_b - \theta_a} \right) (\cos \theta_b - \cos \theta_a). \quad (A-13) \]

Finally, substituting the value of \((s_b - s_a)\) from Eq. A-11 into Eq. A-13,
\[ x_b - x_a = \left( \frac{L_b - L_a}{\theta_b - \theta_a} \right) (\cos \phi_a - \cos \phi_b)(\cos \theta_a - \cos \theta_b) \]
\[ \frac{\theta_b - \theta_a}{s_b - s_a} \]
\[ \ldots \ldots \ldots \ldots \quad (A-14) \]

Eq. A-14 gives the true surface displacement along the east-west direction (east is positive). To obtain that along the north-south direction we again write from calculus
\[ \frac{d\theta}{ds} = \frac{dy}{ds} \frac{d\theta}{dy} \quad \ldots \ldots \ldots \quad (A-15) \]

\[ \frac{b}{a} \]
\[ \frac{\theta_b - \theta_a}{s_b - s_a} \]
\[ \text{Separating variables and integrating,} \]
\[ \int_a^b dy = \left( \frac{s_b - s_a}{\theta_b - \theta_a} \right) \int \cos \theta d\theta \]
\[ y_b - y_a = \frac{s_b - s_a}{\theta_b - \theta_a} (\sin \theta_b - \sin \theta_a). \quad (A-16) \]

Substitution of the expression for \((s_b - s_a)\) found in Eq. A-11 into Eq. A-16 gives
\[ y_b - y_a = \left( \frac{L_b - L_a}{\theta_b - \theta_a} \right) (\cos \phi_a - \cos \phi_b)(\sin \theta_a - \sin \theta_b) \]
\[ \frac{\phi_b - \phi_a}{\theta_b - \theta_a} \]
\[ \ldots \ldots \ldots \ldots \quad (A-17) \]

which is the north-south true displacement (north is positive).

Before we complete these equations, we must consider those instances in which the change in either drift angle or its azimuth is zero. That is, \(\phi_b = \phi_a\), or \(\theta_b = \theta_a\). If both these conditions exist, we know intuitively that the Curvature method must yield the equations of the Tangential method.

Before proceeding, therefore, let us prove two equalities.

\[ \lim_{\theta_b \to \theta_a} \left( \frac{\sin \theta_b - \sin \theta_a}{\theta_b - \theta_a} \right) = \cos \theta_a \quad \text{(A-18)} \]
\[ \lim_{\theta_b \to \theta_a} \left( \frac{\cos \theta_b - \cos \theta_a}{\theta_b - \theta_a} \right) = \sin \theta_a \quad \text{(A-19)} \]

To prove these indeterminate forms, let \(\theta_a\) be constant and let \(\theta_b\) approach it. Then, by L'Hospital theorem,
\[ \lim_{\theta_b \to \theta_a} \left( \frac{\sin \theta_b - \sin \theta_a}{\theta_b - \theta_a} \right) = \lim_{\theta_b \to \theta_a} \left[ \frac{d}{d\theta_b} \left( \frac{\sin \theta_b - \sin \theta_a}{\theta_b - \theta_a} \right) \right] = \cos \theta_a, \]
which proves Eq. A-18. Similarly in Eq. A-19,
\[ \lim_{\theta_b \to \theta_a} \left( \frac{\cos \theta_b - \cos \theta_a}{\theta_b - \theta_a} \right) = \sin \theta_a \]

which proves Eq. A-19. We may use Eqs. A-18 and A-19 to simplify Eqs. A-4, A-14 and A-17. Thus, using these generalized expressions, we may write solutions to all four cases shown in the text.

**APPENDIX B**

**Vertical Thickness Correction Equation**

Let us represent the top of the bed by a plane in space with a general equation:
\[ Ax + By + Cz + D = 0. \quad \ldots \ldots \ldots \quad (B-1) \]

If the origin \((0, 0, 0)\) is located on this plane, it can be shown that \(D = 0\) and Eq. B-1 reduces to
\[ Ax + By + Cz = 0 \quad \ldots \ldots \ldots \quad (B-2) \]

The bottom of the bed may also be represented by a plane parallel to the top one. It must satisfy the condition that
\[ A = B = C = 0 \quad \ldots \ldots \ldots \quad (B-3) \]

We may write the equation of a particular plane by making Eq. B-3 equal to unity. Thus
\[ \begin{align*}
A &= A' \\
B &= B' \\
C &= C' \\
\end{align*} \quad \ldots \ldots \ldots \quad (B-4) \]

If this condition exists, and the second plane does not coincide with the first, its generalized equation must be that shown on Eq. B-1 with \(D \neq 0\).

The vertical distance separating these two planes may be computed by solving for the \(z\) intercept of the second plane. Thus, for \(x = 0\) and \(y = 0\), \(Cz + D = 0\) and
\[ z = \frac{-D}{C} \quad \ldots \ldots \ldots \quad (B-5) \]

To apply these concepts to true vertical thickness of a bed pierced by a directional well, we may translate the origin to that point where the well penetrates the top of the bed. With a knowledge of the dip and strike of the bed and the drilled thickness, we can determine the equation of the top plane and solve for that of the bottom plane.

With a previous knowledge of the spatial location of points above and below the bed, obtained from the curvature analysis, we may obtain the true coordinates of the points \(o\) and \(d\) at which the well penetrates the top and bottom of the bed, respectively.

Let us consider Fig. 4, where directional survey points \(o\) and \(d\) are above and below the bed boundaries, respectively. Since by definition the segment from \(a\) to \(b\) has a constant curvature, we may write
\[ \frac{L_b - L_a}{\phi_b - \phi_a} = \frac{L_a - L_o}{\phi_o - \phi_a} \quad \ldots \ldots \ldots \quad (B-6) \]

or
Fig. 4—Representation of two parallel planes in space.

\[ \phi_o = \frac{(L_o - L_a)(\phi_b - \phi_o)}{(L_b - L_a)} + \phi_o \quad \text{(B-7)} \]

Similarly,

\[ \phi_d = \frac{(L_d - L_a)(\phi_b - \phi_o)}{(L_b - L_a)} + \phi_o \quad \text{(B-8)} \]

The expressions for the location of points \( o \) and \( d \) are similar to those given in the text as Eqs. 1 through 10, with the subscripts \( a \) and \( b \) replaced by the subscripts \( o \) and \( d \).

The equation of the top plane is determined by manipulating the dip angle \( \omega \) and its azimuth \( \gamma \) to determine three points. From Fig. 5,

Point 1: \( x_1 = 0, y_1 = 0, z_1 = 0 \)

Point 2: \( x_2 = \sin \gamma \)
  \( y_2 = \cos \gamma \)
  \( z_2 = \tan \alpha_t \)

Point 3: \( x_3 = \sin (\gamma + 90^\circ) \)
  \( y_3 = \cos (\gamma + 90^\circ) \)
  \( z_3 = 0 \)

By substituting these values into Eq. B-2 and solving, we may find expressions for \( A \), \( B \), and \( C \).

\[ A = \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} = y_2 z_3 - y_3 z_2 = -\cos (\gamma + 90^\circ) \tan \alpha_t \quad \text{(B-9)} \]

\[ B = \begin{vmatrix} z_2 & x_2 \\ z_3 & x_3 \end{vmatrix} = z_2 x_3 - z_3 x_2 = \tan \alpha_t \sin (\gamma + 90^\circ) \quad \text{(B-10)} \]

\[ C = \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = x_2 y_3 - x_3 y_2 = \sin \gamma \cos (\gamma + 90^\circ) - \sin (\gamma + 90^\circ) \cos \gamma \quad \text{(B-11)} \]

Since we know the location of one point on the bottom plane, we may determine \( D \) in Eq. B-1 thus:

\[ x' = x_d - x_o \quad \text{(B-12)} \]
\[ y' = y_d - y_o \quad \text{(B-13)} \]
\[ z' = z_d - z_o \quad \text{(B-14)} \]

Solving for \( D \) in Eq. B-1 and substituting the values found for \( x', y', \) and \( z' \), we obtain

\[ D = -(Ax' + By' + Cz') \]

\[ D = -((x_d - x_o) \cos (\gamma + 90^\circ) \tan \omega + (y_d - y_o) \tan \alpha_t \sin (\gamma + 90^\circ) + (z_d - z_o) \cdot \sin \gamma \cos (\gamma + 90^\circ) - \sin (\gamma + 90^\circ) \cos \gamma) \] \quad \text{(B-15)}

And substituting Eqs. B-15 and B-11 into Eq. B-5 yields the expression for the corrected vertical thickness:

\[ h_t = (z_d - z_o) + ((\tan \alpha_t [(y_d - y_o) \sin (\gamma + 90^\circ) - (x_d - x_o) \cos (\gamma + 90^\circ)]) + (\sin \gamma \cos (\gamma + 90^\circ) - \sin (\gamma + 90^\circ) \cos \gamma)) \quad \text{(B-16)} \]

Using the formulas for sums of angles, it is possible to reduce Eq. B-16 to that given in the text as Eq. 11.

For the sake of simplicity, it was assumed that the top and bottom of the formation were parallel. This condition is not necessary for the application of Eq. 11, and only the dip and strike of the bottom of the bed must be known. As long as the translated origin is located on the top plane, its dip and strike are not significant, since the analysis could have been based on an imaginary plane parallel to the bottom of the bed.

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